Data-driven delay identification with SINDy

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Abstract. In this work, we investigate the capabilities of the *Sparse Identification of Nonlinear Dynamics* method for time-delay identification. A possible solution is shown how delayed terms can be introduced into the method. We test the robustness and effectiveness of the method through data generated by simulation of different reference systems with known time delay. Through our test examples, we investigate the effect of noise and the delay distribution in the candidate terms. We also test the method in the presence of multiple delays. It is shown that by iterating through a range of threshold values with the STLSQ algorithm, the delayed terms can be identified in a robust manner.

Keywords: Scientific Machine Learning, Sparse Identification of Nonlinear Dynamics, Time-delay identification

1 Introduction

Model construction based on data-driven techniques has gained considerable ground over the past years due to versatile measurement tools (such as image recognition, smartphone sensors, etc.) and a large amount of available data. The development of computer sciences, statistical and machine learning tools enable more efficient data processing and model discovery. The appearance of Scientific Machine Learning [1] (SciML) made the model construction to be more sophisticated by opening a physics-informed toolset for researchers to discover unexplained phenomena in dynamic systems.

Although discovering governing equations of physical systems is a challenge, the wide availability of computers made it possible to have an extensive amount of data in our possession. Together with the increasing computational power opened the door for us to use data-driven techniques for model construction. Fitting linear models with data-driven techniques has been possible with dynamic mode decomposition [4] but constructing models with nonlinear structure is even a challenge today.

Throughout this study we investigate the *Sparse Identification of Nonlinear Dynamics* (SINDy) method. The core idea of the method was first presented by Brunton et al. in 2016 [2] to explore the nonlinear equations that describe the underlying physics of a dynamic system. Namely, the data driven SINDy can be used to identify the key nonlinearities that may be implemented in the differential equations to describe the properties of the system. In this work, we extend the SINDy algorithm to find the underlying dynamics of systems with time delays. We investigate the limits and the prerequisites of this technique through numerical experiments using a dataset constructed by simulating dynamic systems with known parameters and time delays. We also include a stochastic effect to test the robustness of the method.

2 Sparse Identification of Nonlinear Dynamics for time delay identification

Let us consider a smooth dynamic system that is described by the differential equation of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x} = \mathbf{f}\left(\mathbf{x}(t), \mathbf{x}(t-\tau_1), \mathbf{x}(t-\tau_2), \ldots, t\right), \tag{1}$$

where $\mathbf{x}$ is the state of the system, $\dot{\mathbf{x}}$ denotes its derivative with respect to time t and $\tau_1, \tau_2, \ldots, \tau_{N_\tau} > 0$ are the time-delays. Suppose that we have measurements or any numerical data about the evolution of the solution in time. We arrange these time-series data of the system variables

$$\underset{m\times s}{\mathbf{X}} = \left[\mathbf{x}(t_1)\ \mathbf{x}(t_2)\ \ldots\ \mathbf{x}(t_m)\right]^{\mathrm{T}}, \tag{2}$$

where $t_1, t_2, \ldots, t_m$ are the (not necessarily equidistant) sampling instances. The parameter s denotes the number of state variables. Similarly, we introduce the shifted state variables of the measurements as

$$\underset{m\times s}{\mathbf{X}_{\tau_j}} = \left[\mathbf{x}(t_1-\tau_j)\ \mathbf{x}(t_2-\tau_j)\ \ldots\ \mathbf{x}(t_m-\tau_j)\right]^{\mathrm{T}}, j = 1, 2, \ldots, N_\tau. \tag{3}$$

We also collect the time derivative of the state variables in

$$\underset{m\times s}{\dot{\mathbf{X}}} = \left[\dot{\mathbf{x}}(t_1)\ \dot{\mathbf{x}}(t_2)\ \ldots\ \dot{\mathbf{x}}(t_m)\right]^{\mathrm{T}}. \tag{4}$$

In case $\mathbf{X}_\tau$ or $\dot{\mathbf{X}}$ are not available through direct measurements, we can obtain them through interpolation or numerical differential schemes respectively.

Based on the core idea of SINDy, we can represent (1) using (2)-(4) as

$$\underset{m\times s}{\dot{\mathbf{X}}} = \underset{m\times p}{\boldsymbol{\Theta}}(\mathbf{X}, \mathbf{X}_{\tau_1}, \mathbf{X}_{\tau_2}, \ldots)\underset{p\times s}{\boldsymbol{\Xi}}, \tag{5}$$

where the matrix $\boldsymbol{\Theta}$ contains the candidate terms that may be used to construct the right hand side of (1) and can be defined arbitrarily. Thus, dimension p is determined by the number of arbitrarily chosen candidate terms.The matrix $\boldsymbol{\Xi}$ determines the coefficients of the arbitrarily chosen candidate terms of $\boldsymbol{\Theta}$, ideally, it selects the terms that compose the differential equation (1). Thus, during the SINDy procedure we determine the coefficient matrix $\boldsymbol{\Xi}$ using an optimization algorithm. We apply the SINDy with the sequentially thresholded least squares algorithm (STLSQ) [3].

To obtain the k-th column $\boldsymbol{\xi}_k$ of the matrix $\boldsymbol{\Xi}$ we need to solve the following optimization problem

$$\underset{p\times 1}{\boldsymbol{\xi}_k} = \text{argmin}_{\boldsymbol{\xi}'_k} \left(\left\| \dot{\mathbf{X}}_k - \boldsymbol{\Theta}(\mathbf{X}, \mathbf{X}_{\tau_1}, \mathbf{X}_{\tau_2}, \ldots)\boldsymbol{\xi}'_k \right\|_2 + \lambda \left\| \boldsymbol{\xi}'_k \right\|_1 \right). \tag{6}$$

The norms are defined as in [3] and λ is a tuneable parameter that affects the sparsity and error tolerance of the dynamical system fitted on the measured data. During our implementation, we introduced an iteration for parameter λ and used the sequential thresholded least-squares (STLSQ) algorithm. In our investigation, we found that we were able to achieve a better fitting solution by favoring sparsity in many cases. In the following sections, we will show how this methodology can contribute to identifying the most prominent terms.

3 Time-delay identification of a system with single delay using SINDy

We take a one dimensional nonlinear delayed system

$$\dot{x}(t) = \alpha x^3(t-\tau) + \alpha x(t-\tau) + \beta \sin(t) \tag{7}$$

and simulate a data-set for given initial conditions and parameters. In our experiment $\tau = 1.0$, $\alpha = -1.0$ and $\beta = 0.09$ while the initial condition is $x(\vartheta) = 0.6$ where $\vartheta \in [-\tau, 0]$.

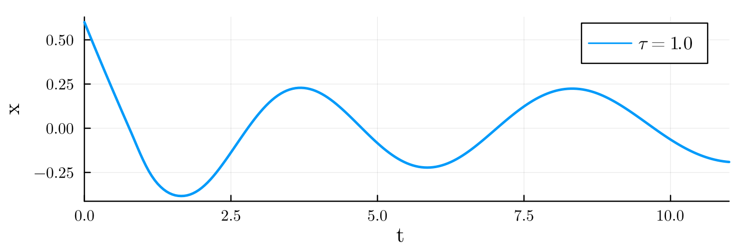

Fig. 1. Solution of (7) for $x(\vartheta) = 0.6$ where $\vartheta \in [-\tau, 0]$ and $\tau = 1.0$, $\alpha = -1.0$ and $\beta = 0.09$. Data-set for the SINDy operation is taken with 0.05 equidistant sampling. We take the samples from $t_0 = 0$ to $t_{220} = 11$.

The main characteristic of the system is provided by the term $\alpha x(t-\tau)$, while additional nonlinearity x^3 and excitation $\sin(t)$ are added in order to test out the sensitivity of the procedure. In order to highlight the capabilities and limitations of this method, the main focus is on the identification of the time-delay in a partially known system. The candidate functions are chosen accordingly, so the algorithm has the chance to capture all details, but we are focused on the delayed state variables. Even with this setup, we can show later the limitations of the method and investigate the effectiveness in identifying the delayed terms. The candidate functions are

$$\underset{221\times 10}{\mathbf{\Theta}}(\mathbf{X}) = \left[\mathbf{1}\ \mathbf{X}\ \mathbf{X}^3\ \mathbf{X}_{1.0}\ \mathbf{X}_{1.0}^3\ \mathbf{X}_{1.5}\ \mathbf{X}_{1.5}^3\ \mathbf{X}_{2.0}\ \mathbf{X}_{2.0}^3\ \sin(\mathbf{t})\right], \tag{8}$$

where the operations (exponentiation, trigonometric function) are interpreted element-wise, meaning that $\mathbf{\Theta}(\mathbf{X})$ has the form

$$\underset{221\times 10}{\mathbf{\Theta}} = \begin{bmatrix} 1 & \mathrm{X}(t_0) & \mathrm{X}^3(t_0) & \mathrm{X}_{1.0}(t_0) & \mathrm{X}_{1.0}^3(t_0) & \mathrm{X}_{1.5}(t_0) & \mathrm{X}_{1.5}^3(t_0) & \mathrm{X}_{2.0}(t_0) & \mathrm{X}_{2.0}^3(t_0) & \sin(t_0) \\ 1 & \mathrm{X}(t_1) & \mathrm{X}^3(t_1) & \mathrm{X}_{1.0}(t_1) & \mathrm{X}_{1.0}^3(t_1) & \mathrm{X}_{1.5}(t_1) & \mathrm{X}_{1.5}^3(t_1) & \mathrm{X}_{2.0}(t_1) & \mathrm{X}_{2.0}^3(t_1) & \sin(t_1) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \mathrm{X}(t_n) & \mathrm{X}^3(t_n) & \mathrm{X}_{1.0}(t_n) & \mathrm{X}_{1.0}^3(t_n) & \mathrm{X}_{1.5}(t_n) & \mathrm{X}_{1.5}^3(t_n) & \mathrm{X}_{2.0}(t_n) & \mathrm{X}_{2.0}^3(t_n) & \sin(t_n) \end{bmatrix}, \tag{9}$$

where $\mathrm{X}_\tau(t) = \mathrm{X}(t-\tau)$ and $n = 220$ was used in our tests. Ideally, if differential equation (7) is represented as (5) with (9), then the coefficient matrix should be

$$\underset{10\times 1}{\mathbf{\Xi}} = \left[0\ 0\ 0\ -\mathbf{1.0}\ -\mathbf{1.0}\ 0\ 0\ 0\ 0\ \mathbf{0.09}\right]^{\mathrm{T}} \tag{10}$$

after the optimization. Applying the SINDy algorithm with a threshold value $\lambda = 0.1$ the identified coefficient matrix is

$$\underset{10\times 1}{\mathbf{\Xi}} = \left[0\ 0\ -0.9\ -\mathbf{1.17}\ -\mathbf{0.12}\ 0\ 0\ 0\ 0\ \mathbf{0}\right]^{\mathrm{T}}, \tag{11}$$

which translates to

$$\dot{x}(t) = -0.9x^3(t) - 1.17x(t-\tau) - 0.12x^3(t-\tau), \tag{12}$$

with $\tau = 1.0$. In Fig. 2, we can see that the identified system follows the characteristic of the original system and the identified delayed terms in (12) are the expected terms. The coefficients are different and there is a dominant misidentified term $x^3(t)$ while the $\sin(t)$ term is completely missing. We can see that the found system is following the main characteristic of the experiment. This can be misleading, since it is a possible outcome of an overfitted system. To im-

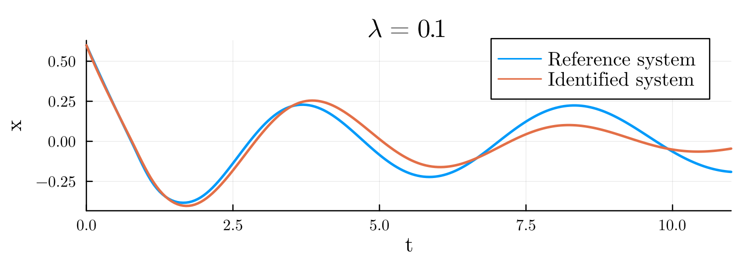

Fig. 2. Result of the SINDy operation on data-set generated with (7) using threshold $\lambda = 0.1$. Identified system (12) is presented along with the reference solution.

prove confidence in the identified system we will use parameter λ to explore the system's main behavior. Namely, there may be terms repeatedly appearing for different lambda values. Hence, we can be certain that those terms are strongly related to the behavior of the original system. In (6), by increasing the value of λ we are promoting the identification of systems with fewer terms. Using an iteration for a range of λ values we can store the mean squared error (MSE $= \frac{1}{n}\sum_{i=0}^{n}\left(x_{\text{ref}}(t_i) - x(t_i)\right)^2$) of the system (by calculating the squared differences of the identified and reference system time histories) for each identified case and we can extract the delayed terms. We can find the best fitting solution that has the least error. Also, we will take the one with the highest λ value. Our goal is to find the optimum solution that has the least amount of error and contains as few terms as possible.

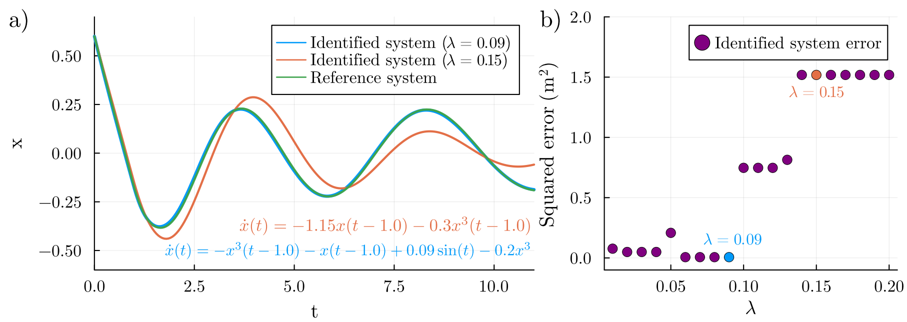

Fig. 3. (a) Two identified systems for different λ threshold values of the STLSQ algorithm. The reference system for this case was (7) with $\tau = 1.0$. By increasing the value of λ we get a system with increased error on behalf of having less terms to describe it. (b) The errors of the identified systems are shown with respect to the λ values. Identification of the best fitting solution is possible this way. Two cases are highlighted with $\lambda = 0.09$ and $\lambda = 0.15$ thresholds in correspondence with panel a.

In Fig. 3(a), the identified systems are shown for the best and worst fitting solutions. The coefficients are presented for these cases along with the reference system in Table 1.

	$x^3(t)$	$x(t-1.0)$	$x^3(t-1.0)$	$\sin(t)$	MSE
Reference	0	-1.0	-1.0	0.09	0
Fit 1 ($\lambda = 0.09$)	-0.2	-1.0	-1.0	0.09	0.006
Fit 2 ($\lambda = 0.15$)	0	-1.15	-0.3	0	1.518

Table 1. Identified coefficient values following the SINDy operation for the best and worst fitting cases with the respective mean squared errors. Reference system is (7) with $\tau = 1.0$. Results are shown visually in Fig.3.

Both systems contain the same delayed terms. The terms that influence the system can be easily identified with the method. On the other hand, with the higher $\lambda = 0.15$ value, we lost details like the time-dependent term $\sin(t)$. With the lower $\lambda = 0.09$ value we caught the details of the system as well but there is a misidentified term x^3 appearing. All in all, the method was able to identify the most prominent terms and the identified delayed terms were correct in both cases.

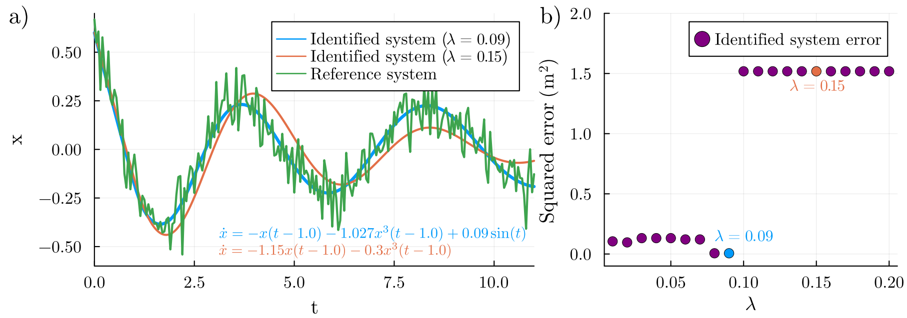

Fig. 4. (a) Two identified systems for different λ threshold values of the STLSQ algorithm. Same reference system as in Fig. 3(a) but with additional random noise with an amplitude of 0.08. (b) The errors of the identified systems are shown with respect to the λ values similar to Fig. 3(b).

In reality, this method may be used for measurement data that is often influenced by measurement noise. Hence, we also added a random noise with an amplitude of 0.08 to the input signal of our data-driven system identification, see Fig. 4. The coefficients are presented for these cases along with the reference system in Table 2.

	$x(t-1.0)$	$x^3(t-1.0)$	$\sin(t)$	MSE
Reference	-1.0	-1.0	0.09	0
Fit 1 ($\lambda = 0.09$)	-1.0	-1.0	0.09	0.006
Fit 2 ($\lambda = 0.15$)	-1.15	-0.3	0	1.518

Table 2. Identified coefficient values following the SINDy operation for the best and worst fitting cases with the respective mean squared errors (compared to the original system solution). Reference system is (7) with $\tau = 1.0$. Dataset is the reference system with added random noise of 0.08. Results are shown visually in Fig.4.

Compared to the previous case the identified system for $\lambda = 0.09$ is more accurate since there is only a small numerical difference in the coefficient of the $x^3(t-\tau)$ term, but the identified terms are the same as in the reference system. In the case of $\lambda = 0.15$, there is no difference compared to the case with no added noise. Additionally, we can see in Fig. 4(b) that the intermediate error showing up in the range from $\lambda = 0.1$ to $\lambda = 0.13$ of Fig. 3 has disappeared. All in all, the noise had no negative impact on the identified system. This is mainly due to the fact that the system kept its main characteristic behavior.

In the previous cases, the base system defined in (8) helped in the identification since (7) shows characteristically different behavior for $\tau = 1.0$, 1.5 and 2.0. Now let us consider the base system

$$\underset{221\times 10}{\mathbf{\Theta}}(\mathbf{X}) = \left[\mathbf{1}\ \mathbf{X}\ \mathbf{X}^3\ \mathbf{X}_{0.9}\ \mathbf{X}_{0.9}^3 \mathbf{X}_{1.0}\ \mathbf{X}_{1.0}^3\ \mathbf{X}_{1.1}\ \mathbf{X}_{1.1}^3\ \sin(\mathbf{t})\right], \quad (13)$$

i.e., select slightly different time delays for the candidate terms. The solutions of (7) for $\tau = 0.9$, 1.0 and 1.1 are shown in Fig. 5. The other parameters are set to $\alpha = 1.0$ and $\beta = 0.09$, as formerly.

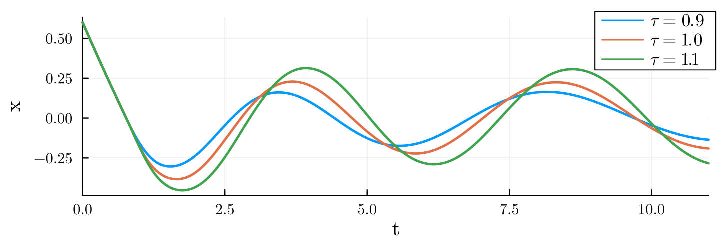

Fig. 5. Numerical solution of (7) for $\tau = 0.9$, 1.0 and 1.1 delays, with the same initial condition.

The behavior of the systems for these three cases are similar and the values at each time point are close. In this case, it is expected that the algorithm will identify a system as an over-fitted combination of the terms in the base system.

Based on Table 3, where the coefficients are presented for these cases along with the reference system, we can conclude that the identified systems are not close to the reference system with this tight delay distribution in the candidate terms.

	$x(t)$	$x^3(t)$	$x_{0.9}(t)$	$x_{0.9}^3$	$x_{1.0}(t)$	$x_{1.0}^3(t)$	$x_{1.1}(t)$	$x_{1.1}^3(t)$	$\sin(t)$	MSE
Reference	0	0	0	0	-1.0	-1.0	0	0	0.09	0
Fit 1 ($\lambda = 0.03$)	0.04	-0.05	-0.9	0.29	0.3	-1.25	-0.5	0.29	0.09	0.089
Fit 2 ($\lambda = 0.16$)	0	0	-0.4	0	-0.6	-2.8	-0.16	2.5	0	0.951

Table 3. Identified coefficient values following the SINDy operation for the best and worst fitting cases with the respective mean squared errors. Reference system is (7) with $\tau = 1.0$. The candidate terms are with a tight delay distribution as (13). Here, $x_\tau^n(t) = x^n(t-\tau)$. Results are shown visually in Fig.6.

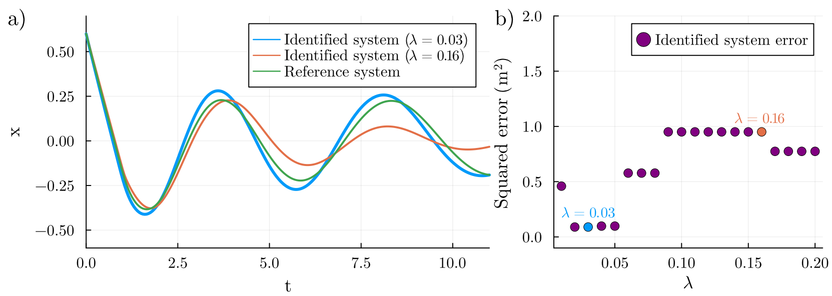

Fig. 6. (a) Two identified systems for the best and worst fitting cases following the SINDy operation. Reference system is (7) with $\tau = 1.0$. In this case, the SINDy operation uses tight delay distribution as the basis. (b) The errors of the identified systems are shown with respect to the λ values similarly to the previous cases.

According to the preliminary expectations, the algorithm constructed a system that is over-fitted to the reference system in this time window.

We can improve the results by expanding the fitted dataset with a simulation from a different initial condition. This way the effect of overfitting can be reduced by providing a broader view of the behavior of the system. The solutions for two different initial conditions are shown in Fig. 7. The dataset provided for the SINDy algorithm is simply constructed by concatenation of the two cases. The resulting coefficients for this case are shown in Table 4.

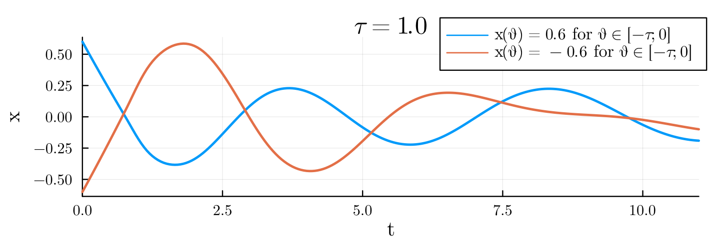

Fig. 7. Solution of (7) with $\tau = 1.0$ from two different initial conditions.

In Fig. 8. the identified system for the best fitting case is still far from the reference system, but with a higher λ value, we were able to identify the proper delayed terms with the cost of losing the time-dependent $\sin(t)$ term. This is a small improvement compared to the case where the dataset was generated based on one initial condition, but it is still not trivial to identify the correct time delay with a tight delay distribution in the base system.

	$x(t-0.9)$	$x(t-1.0)$	$x^3(t-1.0)$	$x(t-1.1)$	$x^3(t-1.1)$	$\sin(t)$	MSE
Reference	0	-1.0	-1.0	0	0	0.09	0
Fit 1 ($\lambda = 0.07$)	-0.6	0	-1.2	-0.4	0.21	0.09	0.078
Fit 2 ($\lambda = 0.26$)	0	-1.0	-1.07	0	0	0	3.532

Table 4. Identified coefficient values following the SINDy operation for the best and worst fitting cases with the respective mean squared errors. Reference system is (7) with $\tau = 1.0$. The candidate terms are with a tight delay distribution as (13) and dataset is based on two ICs. Results are shown visually in Fig.8.

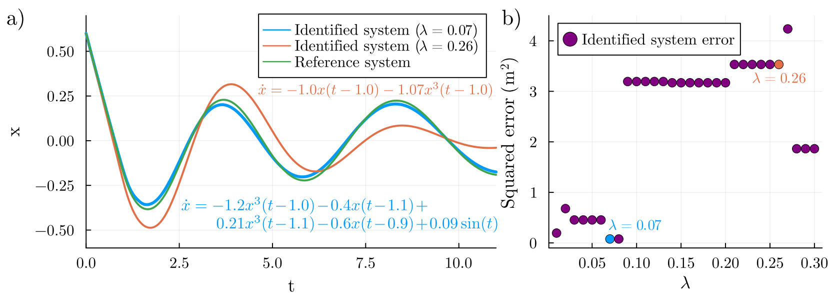

Fig. 8. (a) Two identified systems for the best and nearly the worst ($\lambda = 0.27$ considered as a single outlier) fitting cases following the SINDy operation. Reference system is (7) with $\tau = 1.0$ ran from two different initial conditions. (b) The errors of the identified systems are shown with respect to the λ values similarly to the previous cases.

It is advised therefore to define a base system with sparse delay distribution. It is worth noting however, that in both cases for the best fitting cases (Table 3 and 4), if we sum the coefficients of the respective terms ($x(t-\tau)$, $x^3(t-\tau)$) we obtain a close value of the original system coefficients.

4 Time-delay identification of a system with multiple delays using SINDy

One of the main advantages of this method is that it is applicable to systems containing multiple delays. We modify (7) and consider two separate delays:

$$\dot{x}(t) = \alpha x(t-\tau_1) + \alpha x^3(t-\tau_2) + \beta \sin(t), \tag{14}$$

where $\alpha = -1.0$, $\beta = 0.09$, $\tau_1 = 1.0$ and $\tau_2 = 0.5$. The dataset is generated based on this system for one initial condition of $x(\vartheta) = 0.6$ for $\vartheta \in [-\max\{\tau_1, \tau_2\}, 0]$. The candidate functions are

$$\underset{221\times 10}{\mathbf{\Theta}}(\mathbf{X}) = \left[\mathbf{1}\ \mathbf{X}\ \mathbf{X}^3\ \mathbf{X}_{0.5}\ \mathbf{X}_{0.5}^3\ \mathbf{X}_{1.0}\ \mathbf{X}_{1.0}^3\ \mathbf{X}_{1.5}\ \mathbf{X}_{1.5}^3\ \sin(\mathbf{t})\right]. \tag{15}$$

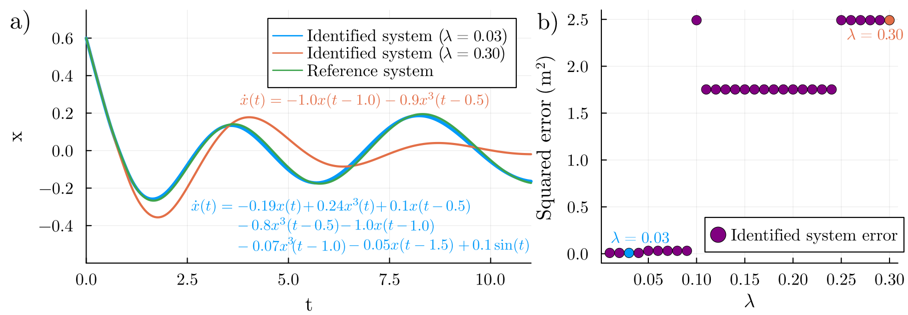

Fig. 9. (a) Two identified systems for the best and the worst fitting cases following the SINDy operation. Reference system is (14) with $\tau_1 = 1.0$ and $\tau_2 = 0.5$. (b) The errors of the identified systems are shown with respect to the λ values similarly to the previous cases.

	$x(t)$	$x^3(t)$	$x_{0.5}(t)$	$x^3_{0.5}$	$x_{1.0}(t)$	$x^3_{1.0}(t)$	$x_{1.5}(t)$	$\sin(t)$	MSE
Reference	0	0	0	-1.0	-1.0	0	0	0.09	0
Fit 1 ($\lambda = 0.03$)	-0.19	0.24	0.1	-0.8	-1.0	-0.07	-0.05	0.1	0.009
Fit 2 ($\lambda = 0.30$)	0	0	0	-0.9	-1.0	0	0	0	2.491

Table 5. Identified coefficient values following the SINDy operation for the best and worst fitting cases with the respective mean squared errors. Reference system is (14) with $\tau_1 = 1.0$ and $\tau_2 = 0.5$. Here, $x^n_\tau(t) = x^n(t-\tau)$. Results are shown visually in Fig.9.

The SINDy algorithm is performed as before. The results can be seen in Fig. 9. The resulting systems in Fig. 9(a) have successfully identified the time-delay. In case of $\lambda = 0.04$ there are a lot of misidentified terms, but the dominant terms are correct for the time delay. The $\lambda = 0.3$ case however was able to identify the correct and only the correct delayed terms while the $\sin(t)$ term was completely missing. All in all, this method is able to identify multiple delays in a single system.

5 Conclusion

We have found that on simulated nonlinear systems, the SINDy algorithm worked well with the proposed method for delay identification. Our main focus was on the identification of the delayed terms. As it can be seen in Table 1, 2, 4 and 5, the delayed terms were properly identified by simultaneously analyzing the best and worst fitting cases. Moreover, if we look at the best fitting cases, we find that the excitation (as term that has smaller effect in our case) was captured in all cases. It should be noted, however, that the focus was not on the construction of the basis during our experiment. We introduced the excitation as a known part of the system. We did not introduce any other candidate terms, so we could focus on the capabilities of the delay identification. In our study, the physically meaningful range of delays had to be determined for the algorithm and the candidate terms had to be introduced with a sparse delay distribution. We experienced difficulties when the delays were closely separated and the found systems were often constructed as a linear combination of those terms. In this case, extracting data related to different initial conditions helped. The developed method is also capable to identify multiple time delays in the same system which was also tested successfully.

Acknowledgement

The research reported in this paper was partly supported by the János Bolyai Research Scholarship of the Hungarian Academy of Sciences and by the National Research, Development and Innovation Office under grant no. NKFI-128422 and under grant no. 2020-1.2.4- TÉT-IPARI-2021-00012.

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